# Danmarks Tekniske Universitet

Written exam, May 13, 2024
Course name: Mathematics 1b
Course number: 01002
Aids: All aids allowed by DTU (without internet)
Duration: 4 hours
Weights: Ex. 1: 20%, Ex. 2: 20%, Ex. 3: 20%, Ex. 4: 20%, Ex. 5: 20%.

*In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish. All references (terminology, definitions, etc.) are to the lecture notes.* A Danish version of the exam set follows *after* the English version.

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**Exercise 1.** Consider the quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ given by

$$q(\boldsymbol{x}) = 5x_1^2 + 5x_2^2 + 8x_3^2 + 8x_1x_2 - 4x_1x_3 + 4x_2x_3 - 22x_1 - 32x_2 - 20x_3 + 53,$$

where $\boldsymbol{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^3$.

(a) Compute the gradient $\nabla q(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^3$.

(b) Compute the Hessian matrix $\boldsymbol{H}_q$. *Hint: The Hessian matrix should not depend on* $\boldsymbol{x}$.

(c) Find an orthonormal basis of eigenvectors for the Hessian matrix $\boldsymbol{H}_q$.

(d) Show that $(1, 2, 1)$ is a stationary point of $q$. Find all stationary points of $q$.

(e) State a direction from the stationary point $(1, 2, 1)$ in which the function $q$ neither increases nor decreases (i.e., a direction where the function is constant).

(f) We now consider the gradient method with learning rate $\alpha = 0.02$ and initial guess $\boldsymbol{x}_0$ given by:

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

Compute $\boldsymbol{x}_{10}$, where

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \alpha \nabla q(\boldsymbol{x}_n), \quad n = 0, 1, 2, \ldots.$$

You should state $\boldsymbol{x}_{10}$ as a vector of *decimal numbers* with an appropriate number of decimals. Which point does the gradient method converge to? (you do not have to provide a proof, just a qualified guess based on your computations).

*The set of problems CONTINUES.*

**Exercise 2.** Consider the quadratic form $q : \mathbb{R}^4 \to \mathbb{R}$ given by

$$q(\boldsymbol{x}) = 2x_1x_3 + 4x_2x_4,$$

where $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4$.

(a) State a *symmetric* matrix $A \in \mathbb{R}^{4\times4}$ such that $q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x}$ , where $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T$.

(b) Find an orthogonal (change-of-basis) matrix $Q \in \mathbb{R}^{4\times4}$ that "reduces" the quadratic form $q$ in the sense that $q$, in the new coordinates, does not contain "mixed terms" of the form $x_ix_j$ (where $i \neq j$). Express $q$ in the new coordinates.

We now consider $q$ restricted to the set

$$B = \{\boldsymbol{x} \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\}.$$

(c) Explain why the function $q : B \to \mathbb{R}$ has a minimal and maximal value.

(d) Determine the minimum value and the maximum value of $q : B \to \mathbb{R}$.

**Exercise 3.** Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \frac{y^2\cos(x)}{x^2+y^2} & \text{for } (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}, \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

(a) Plot the graph of the function $f$.

(b) Compute the two first-order partial derivatives of $f$ for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

(c) Find the degree-two Taylor polynomial $P_2(x)$ of $\cos(x)$ at $x_0 = 0$.

(d) Show that

$$\lim_{x\to0} f(x, x) = \frac{1}{2}.$$

(e) Determine the limit

$$\lim_{x\to0} f(x, 2x).$$

(f) Argue that $f$ is differentiable on $\mathbb{R}^2 \setminus \{(0, 0)\}$, but not in $(0, 0)$.

*The set of problems CONTINUES.*

**Exercise 4.** Consider the vector field $\boldsymbol{V} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\boldsymbol{V}(x, y, z) = (-x, xy^2, x + z)$$

and the curve $\mathcal{K}_1$ given by the parametrization:

$$\boldsymbol{r}(u) = (u, u^2, u + 1), \quad u \in [0, 2].$$

Thus, $\mathcal{K}_1 = \mathrm{im}(\boldsymbol{r})$.

(a) Find the tangent vector $\boldsymbol{r}'(u)$ and argue that the parameterization is regular.

(b) Calculate $\langle \boldsymbol{V}(\boldsymbol{r}(u)), \boldsymbol{r}'(u) \rangle$ for all $u \in [0, 2]$ and compute the line integral $\int_{\mathcal{K}_1} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{s}$.

(c) Find a parametrization $\boldsymbol{p} : [0, 1] \to \mathbb{R}^3$ of the straight line from $(0, 0, 1)$ to $(2, 4, 3)$. We denote the line segment by $\mathcal{K}_2 = \mathrm{im}(\boldsymbol{p})$. Compute the line integral $\int_{\mathcal{K}_2} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{s}$.

(d) Determine if $\boldsymbol{V}$ is a gradient field.

**Exercise 5.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_1 + 1$$

and the subset $A \subset \mathbb{R}^2$ given by:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid -2 \le x_1 \le 2 \wedge -1 \le x_2 \le 1\}.$$

(a) Compute the integral $\int_A f(x_1, x_2)\, \mathrm{d}(x_1, x_2)$.

(b) Determine the volume of the set

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in A \wedge 0 \le x_3 \le f(x_1, x_2)\}.$$

Let $a > 0$. Let $B \subset \mathbb{R}^2$ denote the circular disc centered at the origin with a radius of $a$:

$$B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le a^2\}.$$

(c) Specify a parametrization of the circular disc $B$ and find the associated Jacobian determinant. *Hint: Polar coordinates.*

(d) Determine the value of $a$ to 3 decimal places such that

$$\int_A f(x_1, x_2)\, \mathrm{d}(x_1, x_2) = \int_B f(x_1, x_2)\, \mathrm{d}(x_1, x_2).$$

*The set of problems is completed.*

# Danmarks Tekniske Universitet

Skriftlig eksamen, 13. maj 2024
Kursusnavn: Matematik 1b
Kursusnummer: 01002
Hjælpemidler: Alle hjælpemidler tilladt af DTU (uden internet)
Varighed: 4 timer
Vægtning: Opg. 1: 20%, Opg. 2: 20%, Opg. 3: 20%, Opg. 4: 20%, Opg. 5: 20%.

*For at opnå fuld point, skal der gives fuldstændige argumenter. Svarene kan gives på engelsk eller dansk. Alle referencer (terminologi, definitioner osv.) er til lærebogsnoterne.*

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**Opgave 1.** Betragt den kvadratiske form $q : \mathbb{R}^3 \to \mathbb{R}$ givet ved

$$q(\boldsymbol{x}) = 5x_1^2 + 5x_2^2 + 8x_3^2 + 8x_1x_2 - 4x_1x_3 + 4x_2x_3 - 22x_1 - 32x_2 - 20x_3 + 53,$$

hvor $\boldsymbol{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^3$.

(a) Udregn gradienten $\nabla q(\boldsymbol{x})$ for ethvert $\boldsymbol{x} \in \mathbb{R}^3$.

(b) Udregn Hesse-matricen $\boldsymbol{H}_q$. *Hint: Hesse-matricen skal ikke afhænge af* $\boldsymbol{x}$.

(c) Find en ortonormal basis af egenvektorer for Hesse-matricen $\boldsymbol{H}_q$.

(d) Vis at $(1, 2, 1)$ er et stationært punkt for $q$. Find alle stationære punkter for $q$.

(e) Angiv en retning fra det stationære punkt $(1, 2, 1)$ langs hvilken funktionen $q$ hverken vokser eller aftager (altså en retning hvor funktionen er konstant).

(f) Vi betragter nu gradientmetoden med læringsrate $\alpha = 0.02$ og startgæt $\boldsymbol{x}_0$ givet ved:

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

Udregn $\boldsymbol{x}_{10}$, hvor

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \alpha \nabla q(\boldsymbol{x}_n), \quad n = 0, 1, 2, \ldots.$$

Du bør angive $\boldsymbol{x}_{10}$ som en vektor af *decimaltal* med et passende antal decimaler. Hvilket punkt konvergerer gradientmetoden mod? (du skal ikke give et bevis, blot gætte ud fra dine undersøgelser).

*Opgavesættet FORTSÆTTER.*

**Opgave 2.** Betragt den kvadratiske form $q : \mathbb{R}^4 \to \mathbb{R}$ givet ved

$$q(\boldsymbol{x}) = 2x_1x_3 + 4x_2x_4,$$

hvor $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4$.

(a) Angiv en *symmetrisk* matrix $A \in \mathbb{R}^{4\times 4}$, således at $q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x}$, hvor $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T$.

(b) Find en ortogonal basisskifte-matrix $Q \in \mathbb{R}^{4\times 4}$, der reducerer den kvadratiske form $q$ således, at $q$ i de nye koordinater ikke indeholder "blandede led" af formen $x_ix_j$ (hvor $i \neq j$). Angiv forskriften for $q$ i de nye koordinater.

Vi betragter nu $q$ som en funktion med definitionsmængde $\mathrm{dom}(q) = B$ givet ved:

$$B = \{\boldsymbol{x} \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\}.$$

(c) Gør rede for at $q : B \to \mathbb{R}$ har en minimums- og maksimumsværdi.

(d) Bestem minimumsværdien og maksimumsværdien af $q : B \to \mathbb{R}$.

**Opgave 3.** Lad funktionen $f : \mathbb{R}^2 \to \mathbb{R}$ være givet ved

$$f(x,y) = \begin{cases} \frac{y^2\cos(x)}{x^2+y^2} & \text{for } (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

(a) Plot grafen for funktionen $f$.

(b) Beregn de to førsteordens partielle afledede af $f$ for $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

(c) Find andengrads Taylor-polynomiet $P_2(x)$ for $\cos(x)$ med udviklingspunkt $x_0 = 0$.

(d) Vis at

$$\lim_{x\to 0} f(x,x) = \frac{1}{2}.$$

(e) Bestem grænsen

$$\lim_{x\to 0} f(x,2x).$$

(f) Argumenter for, at $f$ er differentiabel på $\mathbb{R}^2 \setminus \{(0,0)\}$, men ikke i punktet $(0,0)$.

*Opgavesættet FORTSÆTTER.*

**Opgave 4.** Betragt vektorfeltet $\boldsymbol{V} : \mathbb{R}^3 \to \mathbb{R}^3$ givet ved

$$\boldsymbol{V}(x, y, z) = (-x, xy^2, x + z)$$

og kurven $\mathcal{K}_1$ givet ved parameterfremstillingen:

$$\boldsymbol{r}(u) = (u, u^2, u + 1), \quad u \in [0, 2].$$

Altså er $\mathcal{K}_1 = \mathrm{im}(\boldsymbol{r})$.

(a) Udregn tangentvektoren $\boldsymbol{r}'(u)$ og argumenter for at parametriseringen er regulær.

(b) Udregn $\langle \boldsymbol{V}(\boldsymbol{r}(u)), \boldsymbol{r}'(u) \rangle$ for ethvert $u \in [0, 2]$ og udregn det tangentielle kurveintegral $\int_{\mathcal{K}_1} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{s}$.

(c) Angiv en parametrisering $\boldsymbol{p} : [0, 1] \to \mathbb{R}^3$ af den rette linje fra $(0, 0, 1)$ til $(2, 4, 3)$. Vi benævner linjestykket $\mathcal{K}_2 = \mathrm{im}(\boldsymbol{p})$. Udregn det tangentielle kurveintegral $\int_{\mathcal{K}_2} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{s}$.

(d) Afgør om $\boldsymbol{V}$ er et gradientfelt.

**Opgave 5.** Betragt funktionen $f : \mathbb{R}^2 \to \mathbb{R}$ givet ved

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_1 + 1$$

samt mængden $A \subset \mathbb{R}^2$ givet ved:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid -2 \leq x_1 \leq 2 \ \wedge \ -1 \leq x_2 \leq 1\}.$$

(a) Beregn integralet $\int_A f(x_1, x_2) \, \mathrm{d}(x_1, x_2)$.

(b) Bestem rumfanget af mængden

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in A \wedge 0 \leq x_3 \leq f(x_1, x_2)\}.$$

Lad $a > 0$. Lad $B \subset \mathbb{R}^2$ betegne cirkelskiven med centrum i origo og radius $a$:

$$B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq a^2\}.$$

(c) Angiv en parametrisering af cirkelskiven $B$ og find den tilhørende Jacobi-determinant. *Hint: Polære koordinater.*

(d) Bestem værdien af $a$ med 3 decimaler, således at

$$\int_A f(x_1, x_2) \, \mathrm{d}(x_1, x_2) = \int_B f(x_1, x_2) \, \mathrm{d}(x_1, x_2).$$

*Opgavesættet er afsluttet.*